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# **Dispersion analysis of multi-modal waves based on the Reassigned Cross-S-Transform**

By

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## Abstract

1  
2 The quantitative description of the propagation of the energy components in a signal is a  
3 fundamental task in signal processing. The correct identification of the components is important  
4 because they carry information about the medium in which they propagate. In this article, I  
5 introduce the Reassigned Cross-S-Transform to estimate dispersion curves from time-varying  
6 signals, a technique based on function maximization principles and time-frequency cross-  
7 correlation. The RCST provides a sharpened and direct estimate of the dispersion curve for  
8 multi-modal propagating waves. Furthermore, the RCST can be used with signals that are not  
9 synchronized with the same initial time. I present examples studying the degradation effects on  
10 the slowness curves due to mode mixture and reflected waves. I illustrate the application of the  
11 RCST technique to artificially generated Ricker wavelets, and to real recordings of seismic  
12 waves generated by an active source. Finally, using 2D numerical simulations of wave  
13 propagating through an elastic layered medium, I solve a mode identification problem by  
14 combining the RCST with a previously proposed technique to extract Rayleigh waves from  
15 signals.

16 **Keywords:** *geometrical dispersion, time-frequency analysis, multi-modal dispersion.*

## 17      **1. Introduction**

18      Analysis of recorded signals is fundamental in the understanding of wave propagation  
19      phenomena. The Fourier analysis has been a classical technique in the field of signal processing,  
20      due to its robust mathematical foundation and ease of application [1]. The Fourier spectrum  
21      provides insight on the energy of the signal as a function of frequency, allowing the  
22      identification of its components. Spectral analysis has been the basis of quantitative analyses and  
23      modeling of wave propagation problems in many fields such as physics, applied mathematics,  
24      and chemistry. However, most signals related to physical processes are *non-stationary*, this is,  
25      their frequency content change with time, and simple Fourier analysis cannot address this  
26      change. That is why many efforts have been devoted in the last decades to develop Time-  
27      Frequency Representations (TFR), being the most popular the Short-Time Fourier Transform [2],  
28      the Continuous Wavelet Transform [3], the Stockwell Transform [4], and the Empirical Mode  
29      Decomposition [5]. However, according to the Heisenberg-Gabor uncertainty principle [2], most  
30      of the TFRs mentioned above are of limited time-frequency resolution, leading in some cases to  
31      misidentification of components and poor readability. One technique that has been proposed to  
32      improve the readability of TFRs consists of *reassigning* the TFR so that the energy is  
33      concentrated at the time-frequency coordinates where the energy itself attains its maxima.  
34      According to Meignen *et al.* [6], this approach was initiated in the late 1970s by Koderá [7], after  
35      which many reassigned TFRs have been proposed for both non-invertible and fully invertible  
36      distributions. A recent comprehensive review on the subject of reassigned TFRs is provided in  
37      [6].

38      Among the different TFRs, the Stockwell Transform (also known as the S-Transform) has  
39      distinguished itself in practical applications because (i) it preserves the absolute phase of the

40 original time-domain signal, (ii) it can be inverted without losing information. These advantages  
41 have been exploited in recent seismological investigations, where the S-Transform has been  
42 successfully implemented for the identification and extraction of surface waves (Rayleigh and  
43 Love) from strong ground motion recordings, by characterizing the polarization characteristics of  
44 the different wave-types in the time-frequency domain [8]. Furthermore, the phase velocity of  
45 the extracted Rayleigh and Love waves was estimated in the time-frequency domain also making  
46 use of the Stockwell Transform [9]. In a previous publication [10], Stockwell himself had  
47 suggested the use of the Cross-Stockwell-Transform (CST) to estimate phase velocity among  
48 two stations, because the product of the two S-Transforms gives directly the phase difference for  
49 each point of the time-frequency space. However, this method gives correct results inasmuch as  
50 the peaks of the two signals are “located” at the same times.

51 In this work, I extend the ideas of Stockwell regarding the estimation of phase velocity from  
52 recorded signals to implement a time-frequency analysis that permits the estimation of the phase  
53 velocities of the waves making use of the reassignment technique. Different reassignment  
54 approaches have been already proposed for the S-Transform in order to improve its resolution,  
55 such as the generalized S-Transform [11] and the Synchrosqueezed S-Transform [12], among  
56 others. In particular Pan *et al.* [13] performed time-frequency cross-correlation analysis of  
57 signals using the Synchrosqueezed S-Transform, which allows the visualization of the change  
58 with time (the evolution) of the cross-correlation between two signals. Furthermore, the  
59 generalized S-Transform has been implemented by Askari *et al.* [14] in a slant stack procedure to  
60 compute dispersion curves of Rayleigh waves in terms of group velocities. Although the  
61 reassignment technique I propose herein indeed improves the resolution of the S-Transform, its  
62 main objective is to provide a Cross-S-Transform *with energy concentrated at the time-shifts*

63 *between two signals*, which directly gives the relation between slowness and frequency, namely,  
64 *the dispersion curve*. Since spreading of energy is one of the difficulties in identifying different  
65 modes in dispersion analysis, especially when the modes are close to each other (in frequency)  
66 (see for instance, Levshin and Panza [15]), a dispersion curve obtained from reassignment could  
67 be useful to the misidentification problem. Furthermore, as I will show in the sequel, the  
68 *Reassigned Cross-S-Transform (RCST)* is basically a mathematical maximization technique  
69 applied to the S-Transform of the signals, and I make no assumptions regarding the physical  
70 processes represented in the signals: no assumptions regarding either the type of waves or the  
71 spatial-temporal distribution of the phase velocity. Thus, the technique can be applied to signals  
72 related to any wave-propagation problem, provided that they are correctly sampled, as any other  
73 signal processing technique requires.

74 In this article I consider a natural application of the reassignment technique, the dispersion  
75 analysis of Rayleigh waves, since they are waves with strong dispersive characteristics.  
76 Multimodal dispersion analysis of Rayleigh waves is a very common tool for in-situ, non-  
77 destructive testing of different types of materials. In near-surface geophysics in particular, a  
78 popular technique used in seismic surveys is the use of the dispersion curve of Rayleigh waves  
79 propagating through the surface, to derive the shear wave velocity of the different soil layers at a  
80 specific site [16]. The Rayleigh waves are recorded at an arrangement of sensors after they are  
81 excited by a blast or by a sledgehammer. However, in the obtained seismic wave field different  
82 types of waves are present (body, Love, reflected, scattered, noise), which can have a detrimental  
83 impact on the estimation of the desired dispersion curve. Different wave types and different  
84 modes of similar frequency arriving at neighboring times can interfere and degrade the accuracy  
85 of the estimated velocities. Addressing these inaccuracies is still a problem of interest in near-

86 surface geophysics, as the correct determination and interpretation of the dispersion curve is a  
87 fundamental step in a successful surface-wave testing campaign [17].

88 In the following sections, I first explain the basic concepts about the S-transform and about  
89 dispersion curves based on measurements of phase velocity. I then derive the mathematical  
90 formulation to reassign the S-transform of a signal, and extend the reassignment ideas to derive  
91 the Reassigned Cross-S-Transform (RCST). Later, I illustrate the applicability of the RCST with  
92 three experiments of artificial dispersive waves: one with a single wave train, a second one with  
93 two dispersive modes propagating in the same direction, and a third one with two modes  
94 propagating in opposite direction. I implement this last synthetic experiment in order to asses the  
95 effect of reflected waves on the RCST. I also analyse a set of real recordings, acquired to be  
96 analysed with the method of Multichannel Analysis of Surface Waves (MASW) [18], and  
97 compare the obtained dispersion curve with estimations from the more classical F-K Transform  
98 technique. Finally, I illustrate the advantages of using recordings of several components  
99 (horizontal and vertical) to identify and separate different modes of surface waves, and in turn, to  
100 improve the estimation of the dispersion curve. For this, I consider a seismic wave field  
101 generated through 2D numerical simulations with the Spectral Element Method, in a layered  
102 velocity structure where several Rayleigh wave modes propagate.

## 103 **2. Dispersion analysis and the S-transform**

104 Dispersion phenomena is observed when the phase velocity of a wave is frequency-dependent. In  
105 seismological processes dispersion is of two types: material dispersion due to viscoelastic  
106 attenuation, and geometric dispersion due to heterogeneity of the medium [19]. Regardless of the  
107 type of dispersion, a frequency-dependent phase velocity can be defined as follows:

$$v_p(f_n) = \frac{d}{t_p}, \quad t_p = t_p(f_n) \quad (1)$$

108 for each frequency  $f_n$ , where  $d$  is the propagation distance (the distance between the two nearby  
 109 stations where the signals are observed/recorded). The parameter  $t_p$  (the time shift) is then the  
 110 physical quantity that remains frequency-dependent. Phase velocity is a convenient mechanical  
 111 parameter to describe a medium, because the time shift  $t_p$  is a quantity that can be simply and  
 112 directly measured, without making any assumptions regarding the physical process.  
 113 Alternatively, a dispersion curve can be expressed in terms of a frequency-dependent phase  
 114 slowness:

$$s_p(f_n) = \frac{t_p(f_n)}{d} \quad (2)$$

115 In this work, I will then focus on estimating  $t_p$  for wave trains present in two signals recorded at  
 116 nearby stations separated by an interstation distance  $d$ . In addition, I express the dispersion  
 117 curves in terms of the phase slowness, because of its simple proportionality [Eq. (2)] with the  
 118 time shift. The time shift is in turn related to the phase shift  $\varphi_s$  by the simple equation:

$$t_p(f_n) = \frac{\varphi_s}{2\pi f_n}, \quad \varphi_s = \varphi_s(f_n) \quad (3)$$

119 where clearly the phase shift is also a frequency-dependent quantity. A classical wave to estimate  
 120  $\varphi_s$  for a propagating monochromatic wave is to find the peaks of the amplitude of the two signals  
 121 and then take the corresponding phase difference. When the signals are composed of several  
 122 overlapping wave trains, finding such peaks with time-domain processing techniques can be a  
 123 problematic task. Here is when a TFR of the signals becomes a more appropriate tool of analysis,  
 124 since the TFRs provide the evolution of the different frequency components of the signals. The  
 125 TFR I adopted for this study is the Stockwell-Transform, which is defined as follows [4]:



$$S(t, f_n) = \int_{-\infty}^{\infty} u(\tau) \frac{|f_n|}{\sqrt{2\pi}} \exp\left[-\frac{(t-\tau)^2 f_n^2}{2}\right] \exp(-2\pi i f_n \tau) d\tau \quad (4)$$

126 where of  $u(t)$  is the signal to be transformed. At each frequency  $f_n$  the S-transform provides a  
 127 time function called a “voice”, which results from filtering the signal so as to isolate the time  
 128 history of only the  $f_n$  component. Since each value of the S-transform is a complex number, it  
 129 can also be expressed in polar form:

$$S(t, f_n) = A \exp(i\varphi), \quad A = A(t, f_n), \quad \varphi = \varphi(t, f_n) \quad (5)$$

130 where  $A(t, f_n)$  is the amplitude of the complex number,  $\varphi$  is the phase, and  $i = \sqrt{-1}$ . Let us note  
 131 that the phase  $\varphi$  provided by the S-transform is the same phase of the signal at the point  $(t, f_n)$   
 132 [4]. One straightforward way to compute the phase-shift between two voices is to identify the  
 133 location (in time) of the amplitude maxima, and then extract the phase difference at those times.  
 134 However, this method cannot be performed in an automatic manner if there are different wave  
 135 trains present in the signal. For signals composed of single wave trains, the method of computing  
 136 the phase difference using the Fourier Transforms of the signals [20] is much more economic,  
 137 and as accurate. Stockwell [4] had proposed a method of computing the phase difference from  
 138 the Cross-S-Transform of the two signals, which is defined as:

$$S_{12}(t, f_n) = S_1 \cdot \bar{S}_2 \quad (6)$$

139 where  $S_1$  and  $S_2$  are the S-Transforms at stations 1 and 2, respectively, and  $\bar{S}_i$  denotes the  
 140 complex conjugate of  $S_i$ . Station 2 is the station at which the waves arrive later. Using the polar  
 141 forms, the phase difference follows directly:

$$S_{12}(t, f_n) = A_1 \cdot A_2 \exp(i\varphi_1 - i\varphi_2) \quad (7)$$

142 However, for this method to give the correct phase difference, the peaks of the amplitude of the  
143 voices of the two S-Transforms should be located at the same time. Besides, if several wave  
144 trains are present in the signal, they need to be treated one at a time, after identification.

### 145 3. Reassignment of the S-transform

146 The idea of reassigning the S-transform consists in concentrating the energy of the TFR around  
147 the coordinates  $(t, f)$  where the energy itself reaches its maximum values. These coordinates are  
148 usually referred to as the time delay and the instantaneous frequency. The *time delay* is identified  
149 as the time of arrival of the peak of the envelope of an analytical signal. The *instantaneous*  
150 *frequency* is the frequency around which the spectral energy is concentrated at a specific time.  
151 These concepts are immediately applied to the peaks of the amplitude of the S-Transform, since  
152 each voice of the S-transform is equivalent to a band-pass filtered analytical signal of the original  
153 time function [10]. Therefore, the time delay  $t_d$  is identified by finding the arrival time of  
154  $\max_t A(t, f_n)$ . Consequently, in the S-Transform domain the time delay is found at the instants  
155 when the derivative of the amplitude is zero:

$$\left. \frac{dA(t, f_n)}{dt} \right|_{t=t_d} = 0 \quad (8)$$

156 Note that the amplitude of a complex number is a positive function, thus there will be no  
157 “negative peaks”. The problem of finding the time delay is then a maximization problem. The  
158 derivative in Eq. (8) can be approximated numerically, however here I prefer to derive analytic  
159 expressions for it. I start by recalling the polar form of the S-Transform [Eq. (5)] and applying  
160 natural logarithm:

$$\ln[S(t, f_n)] = \ln[A(t, f_n)] + i\varphi(t, f_n) \quad (9)$$

161 Taking the *time* derivative of the logarithmic expression, the following expression is obtained:

$$\frac{1}{S(t, f_n)} \frac{dS(t, f_n)}{dt} = \frac{1}{A(t, f_n)} \frac{dA(t, f_n)}{dt} + i \frac{d\varphi(t, f_n)}{dt} \quad (10)$$

162 In view of the fact that  $A(t, f_n)$  &  $\varphi(t, f_n)$  are *real* quantities, it is true that:

$$\frac{1}{A(t, f_n)} \frac{dA(t, f_n)}{dt} = \mathcal{R}e \left\{ \frac{1}{S(t, f_n)} \frac{dS(t, f_n)}{dt} \right\} \quad (11)$$

163 where  $\mathcal{R}e\{\cdot\}$  denotes the real part of its argument. Starting from the definition of S-Transform

164 given in Eq. (4), it is straightforward to derive an expression for the derivative  $dS(t, f_n)/dt$ :

$$\begin{aligned} \frac{dS(t, f_n)}{dt} &= - \int_{-\infty}^{\infty} u(\tau) \frac{|f_n|}{\sqrt{2\pi}} \exp \left[ -\frac{(t-\tau)^2 f_n^2}{2} \right] \exp[-2\pi i f_n \tau] (t-\tau) f_n^2 d\tau \\ &= -t f_n^2 S(t, f_n) + f_n^2 S_t(t, f_n) \end{aligned} \quad (12)$$

165 where  $S_t(t, f_n)$  is the S-Transform of the product  $t \cdot u(t)$ :

$$S_t(t, f_n) = \int_{-\infty}^{\infty} \tau u(\tau) \frac{|f_n|}{\sqrt{2\pi}} \exp \left[ -\frac{(t-\tau)^2 f_n^2}{2} \right] \exp[-2\pi i f_n \tau] d\tau \quad (13)$$

166 From re-arranging expression (12) above, I obtain the expression:

$$\frac{1}{S(t, f_n)} \frac{dS(t, f_n)}{dt} = f_n^2 \left[ \frac{S_t(t, f_n)}{S(t, f_n)} - t \right] \quad (14)$$

167 Using the latter result in Eq. (11) gives:

$$\frac{dA(t, f_n)}{dt} = f_n^2 \left( \mathcal{R}e \left\{ \frac{S_t(t, f_n)}{S(t, f_n)} \right\} - t \right) A(t, f_n) \quad (15)$$

168 Consequently, the analytical expression for the time delay is obtained:

$$\left. \frac{dA(t, f_n)}{dt} \right|_{t=t_d} = 0 \Rightarrow t_d(f_n) \stackrel{\text{def}}{=} \mathcal{R}e \left\{ \frac{S_t(t_d, f_n)}{S(t_d, f_n)} \right\} \quad (16)$$

169 The time delay computed with Eq. (16) requires the computation of two transforms, the S-  
 170 Transform of  $u(t)$  and the S-transform of  $t \cdot u(t)$ . An analytical expression can be also derived  
 171 for the *Instantaneous Frequency*, which may be defined as follows [10]:

$$IF(t) \stackrel{\text{def}}{=} \frac{1}{2\pi} \frac{\partial}{\partial t} \{2\pi f_n t + \varphi(t, f_n)\} \quad (17)$$

172 From Eq. (10) is evident that:

$$\frac{d\varphi(t, f_n)}{dt} = \mathcal{Jm} \left\{ \frac{1}{S(t, f_n)} \frac{dS(t, f_n)}{dt} \right\} \quad (18)$$

173 where  $\mathcal{Jm}\{\cdot\}$  denotes the imaginary part of its argument. Using Eq. (14) the result for the  
 174 Generalized Instantaneous Frequency is obtained:

$$GIF(t, f_n) = f_n + \left( \frac{f_n^2}{2\pi} \right) \mathcal{Jm} \left\{ \frac{S_t(t, f_n)}{S(t, f_n)} \right\} \quad (19)$$

175 Similarly, a Generalized time-frequency distribution of the time delay can be defined as:

$$T_d(t, f_n) \stackrel{\text{def}}{=} \mathcal{Re} \left\{ \frac{S_t(t, f_n)}{S(t, f_n)} \right\} \quad (20)$$

176 which is then used to reassign the S-Transform through the sifting effect of convolving  $S(t, f_n)$   
 177 with the Delta-Dirac function:

$$S_R(T_d, f_n) = \int_{-\infty}^{\infty} S(\tau, f_n) \delta(T_d - \tau) d\tau \quad (21)$$

178 What the reassignment means is that the S-transform is now evaluated at the coordinates  $(T_d, f_n)$ ,  
 179 the energy is concentrated in the neighboring of the time-delay  $t_d$ , and thus the localization of  
 180 energy is more precise. Now, in order to obtain a dispersion curve I am not interested in  
 181 concentrating the energy around  $t_d$ , the time delay at each station, but around  $t_p$ , *the time shift*

182 *between the two stations* [given in Eq. (2)]. This will provide the variation of  $t_p$  with frequency,  
 183 without the need of finding any maxima, and all waves present in the signal are automatically  
 184 considered. Thus, instead of reassigning each individual S-transform, I want to *reassign the*  
 185 *Cross-S-Transform* [as given in Eq. (6)].

#### 186 **4. Reassignment of the Cross-S-Transform**

187 Following the ideas of the reassigned S-transform presented in the previous section, I want to  
 188 reassign the Cross-S-Transform (CST) on a time-frequency distribution of the time-shift, that I  
 189 denote here by  $T_p$ , in the following manner

$$S_{RC}(T_p, f_n) = \int_{-\infty}^{\infty} S_C(\tau, f_n) \delta(T_p - \tau) d\tau \quad (22)$$

190 and using a redefined Cross-S-Transform:

$$S_C(t, f_n) = S_1 \cdot \bar{S}_{2s} \quad (23)$$

191 where now  $S_1$  is multiplied by the shifted S-Transform of station 2,  $S_{2s}$ . More precisely,  $S_{2s}$  is  
 192 the result of shifting each voice of  $S_2$ , so that the amplitude peaks of  $S_1$  and  $S_{2s}$  are located at the  
 193 same times. Because the voices are functions of time, the delay between two voices is simply  
 194 obtained by applying time-domain cross-correlation to each pair of voices:

$$t_l(f_n) = \operatorname{argmax}_t \int_{-\infty}^{\infty} S_1(\tau, f_n) S_2(\tau + t, f_n) d\tau \quad (24)$$

195 And then each voice  $S_2(\tau, f_n)$  is shifted  $t_l$  seconds to obtain  $S_{2s}(\tau, f_n)$ . The effect of a shift in  
 196 time of the S-Transform becomes evident when expressing the S-Transform as the inverse  
 197 Fourier Transform of a convolution [10]:

$$S(\tau, f_n) = \int_{-\infty}^{\infty} U(\alpha + f_n) \frac{|f_n|}{\sqrt{2\pi}} \exp\left[-\frac{2\pi^2\alpha^2}{f_n^2}\right] \exp(2\pi i\alpha\tau) d\alpha \quad (25)$$

198 Letting  $\tau = t - t_l(f_n)$

$$\begin{aligned} S(t - t_l, f_n) &= \exp(-2\pi i\alpha t_l) \int_{-\infty}^{\infty} U(\alpha + f_n) \frac{|f_n|}{\sqrt{2\pi}} \exp\left[-\frac{2\pi^2\alpha^2}{f_n^2}\right] \exp(2\pi i\alpha t) d\alpha \\ &= \exp(-2\pi i\alpha t_l) S(t, f_n) \end{aligned} \quad (26)$$

199 Therefore,  $S_{2s}(t, f_n)$  can be obtained by simply multiplying each voice of  $S_2(t, f_n)$  by the  
 200 corresponding complex exponential  $\exp(2\pi i\alpha t_l)$ , where  $t_l = t_l(f_n)$ . At this point I would like to  
 201 remark that the shifted transform  $S_{2s}(t, f_n)$  will be the same even if the signal at station 2 does  
 202 not have the same initial time at station 1. With different initial times the parameter  $t_l(f_n)$  will be  
 203 different from the case of having the same initial time, but not the final result  $S_{2s}(t, f_n)$ . And  
 204 thus the technique I propose herein can be used to compute phase velocities even for data that is  
 205 not synchronized with the same initial time.

206 In what follows, I present how the expression for  $T_p$  can be derived,  $T_p$  being the Generalized  
 207 Time-Shift once the two S-Transforms have the peaks at the same coordinates. I start by  
 208 recognizing the analogy between the time delay and the time shift: the time delay gives the  
 209 localization in time of each amplitude peak of the S-transform, and the time shift is associated to  
 210 peaks of the CST. In both cases a maximization is made, that is, time values are found such that  
 211 derivatives are zero. Since the CST is also a complex number, it can be expressed in polar form:

$$S_C(t, f_n) = A_C \exp(i\varphi_d), \quad A_C = A_C(t, f_n), \quad \varphi_d = \varphi_d(t, f_n) \quad (27)$$

212 where  $\varphi_d$  is the phase difference between  $S_1$  and  $S_{2s}$ . From Eq. (23) I recall that the CST is the  
 213 product of two complex-valued functions, its derivative then is given by:

$$\frac{dS_C(t, f_n)}{dt} = S_1(t, f_n) \frac{d\bar{S}_{2s}(t, f_n)}{dt} + \frac{dS_1(t, f_n)}{dt} \bar{S}_{2s}(t, f_n) \quad (28)$$

214 Using Eq. (14) the following expressions are obtained:

$$S_1(t, f_n) \frac{d\bar{S}_{2s}(t, f_n)}{dt} = S_1(t, f_n) [-t f_n^2 \bar{S}_{2s}(t, f_n) + f_n^2 \bar{S}_{2st}(t, f_n)] \quad (29)$$

$$\frac{dS_1(t, f_n)}{dt} \bar{S}_{2s}(t, f_n) = [-t f_n^2 S_1(t, f_n) + f_n^2 S_{1t}(t, f_n)] \bar{S}_{2s}(t, f_n)$$

215 These results can then be substituted in Eq. (28):

$$\frac{dS_C(t, f_n)}{dt} = -2t f_n^2 S_C(t, f_n) + f_n^2 [S_1(t, f_n) \bar{S}_{2st}(t, f_n) + S_{1t}(t, f_n) \bar{S}_{2s}(t, f_n)] \quad (30)$$

216 Thus the derivative of the CST can be found by evaluating four S-transforms. Now, different  
 217 from the case of the time delay, I will not focus here on finding a time when the derivative of an  
 218 amplitude is zero as in [Eq. (16)]. Here the goal is to find the phase shift *when the total*  
 219 *derivative of the CST is zero*. To show that this is the case I go back to Eq. (27) and take the  
 220 logarithm

$$\ln S_C(t, f_n) = \ln A_C(t, f_n) + i\varphi_d(t, f_n) \quad (31)$$

221 The derivative will then be:

$$\frac{1}{S_C(t, f_n)} \frac{dS_C(t, f_n)}{dt} = \frac{1}{A_C(t, f_n)} \frac{dA_C(t, f_n)}{dt} + i \frac{d\varphi_d(t, f_n)}{dt} \quad (32)$$

222 It is then clear that enforcing a zero total derivative implies not only that the derivative of the  
 223 amplitude of the CST at some time  $t_s$  is zero, but also that the derivative of its phase is zero at  $t_s$ .

$$\left. \frac{d\varphi_d(t, f_n)}{dt} \right|_{t=t_s} = 0 \quad (33)$$

224 The underlying assumption is then

$$\frac{d\varphi_1(t_s, f_n)}{dt} = \frac{d\varphi_2(t_s, f_n)}{dt} \quad (34)$$

225 To understand that this is the case, let me recall that the peaks of  $S_1(t, f_n)$  and  $S_{2s}(t, f_n)$  are  
 226 located at the same  $(t, f)$  coordinates, which is equivalent to say that  $S_1(t, f_n)$  and  $S_{2s}(t, f_n)$   
 227 have *the same time delay*  $t_d$  and *the same instantaneous frequency*. Thus, from the definition of  
 228 IF given in Eq. (17) it is evident that relation (34) must hold. Furthermore, if  $t_s = t_d$ , the  
 229 derivatives of the amplitudes  $A_1$  and  $A_2$  are zero at  $t_s$ , and consequently, the derivative of  $A_c$  at  
 230  $t_s$  must be zero too. I now proceed to enforce a zero-derivative of the CST:

$$\begin{aligned} \frac{dS_C(t_s, f_n)}{dt} &= -2t_s f_n^2 S_C(t_s, f_n) + f_n^2 [S_1(t_s, f_n) \bar{S}_{2st}(t_s, f_n) + S_{1t}(t_s, f_n) \bar{S}_{2s}(t_s, f_n)] \quad (35) \\ &= 0 \end{aligned}$$

231 Solving for  $S_C(t_s, f_n)$ , the result is the following

$$2tS_C(t_s, f_n) = [S_1(t_s, f_n) \bar{S}_{2st}(t_s, f_n) + S_{1t}(t_s, f_n) \bar{S}_{2s}(t_s, f_n)] \quad (36)$$

232 Since  $S_C(t_s, f_n) = A_C(t_s, f_n) \exp[i\varphi_d(t_s, f_n)]$  the latter expression becomes:

$$2tA_C(t_s, f_n) \exp[i\varphi_d(t_s, f_n)] = [S_1(t_s, f_n) \bar{S}_{2st}(t_s, f_n) + S_{1t}(t_s, f_n) \bar{S}_{2s}(t_s, f_n)] \quad (37)$$

233 Now taking the natural logarithm

$$\ln 2tA_C(t_s, f_n) + i\varphi_d(t_s, f_n) = \ln [S_1(t_s, f_n) \bar{S}_{2st}(t_s, f_n) + S_{1t}(t_s, f_n) \bar{S}_{2s}(t_s, f_n)] \quad (38)$$

234 Solving for  $\varphi_d$  I then obtain the phase shift under the maximization conditions:

$$\varphi_d(t_s, f_n) = \mathcal{Jm} \{ \ln [S_1(t_s, f_n) \bar{S}_{2st}(t_s, f_n) + S_{1t}(t_s, f_n) \bar{S}_{2s}(t_s, f_n)] \} \quad (39)$$

235 And from this results I defined a TFR which I call the Generalized Phase Shift:



$$\Phi_d(t, f_n) = \mathcal{I}m \{ \ln[S_1(t, f_n)\bar{S}_{2st}(t, f_n) + S_{1t}(t, f_n)\bar{S}_{2s}(t, f_n)] \} \quad (40)$$

236 Finally, with substitution of this result in Eq. (3) I obtain a TFR for the time shift, that is, the

237 Generalized Time Shift:

$$T_p(t, f_n) = \frac{1}{2\pi f_n} \mathcal{I}m \{ \ln[S_1(t, f_n)\bar{S}_{2st}(t, f_n) + S_{1t}(t, f_n)\bar{S}_{2s}(t, f_n)] \} \quad (41)$$

238 With this key result I can re-assign the Cross-S-Transform, using Eq. (22). In the next sections I

239 illustrate how all these theoretical derivations can be used to obtain dispersion curves.

## 240 **5. Example with one artificial dispersive wave train**

241 In this example, I propagate a Ricker wavelet through different stations, imposing an artificial

242 dispersion curve. The Ricker wavelet is given by the expression:

$$R(t) = A_R(2\beta - 1) \exp(-\beta), \quad \beta = [\pi f_c(t - t_c)]^2 \quad (42)$$

243 where  $A_R$  is the amplitude and  $t_c$  is the time of its largest peak. The Ricker wavelet has a limited

244 bandwidth with energy concentrated at its central frequency  $f_c$ . This limited bandwidth is one of

245 the reasons why the Ricker wavelet is used in many seismological applications, since registered

246 seismograms are usually composed of several wave trains of limited duration and limited

247 frequency band. To propagate this wavelet with frequency-dependent phase velocity I adopt the

248 following artificial dispersion curve:

$$V_p = 1.7 \exp\left(-\frac{f}{15}\right) + 0.4 \left(\frac{km}{s}\right) \quad (43)$$

249 In Figure 1 I plot this dispersion curve given in terms of phase velocity, and most importantly, in

250 terms of the slowness. With this artificial slowness, I propagate the dispersive Ricker wavelet at

251 20 stations, with a uniform interstation distance  $\Delta x$  of 5 m (typical in geophysical

252 measurements). For the Ricker wavelet I choose  $A_R = 1$ ,  $f_c = 30$  Hz, and  $t_c = 1/f_c$ . In order to

253 have precise measurements of small time shifts, I select the time step  $\Delta t$  to be  $1.43\text{E-}4$  s. For  
 254 each discrete frequency, I apply the time shift to the Ricker wavelet in the frequency domain via  
 255 the Fourier Transform and convolution. Thus, the waveform at some station  $j + 1$  is obtained  
 256 from the waveform at station  $j$ :

$$R_{j+1}(t) = F^{-1}\{F\{R_j(t)\} \cdot \exp(-2i\pi f s_p \Delta x)\} \quad (44)$$

257 where  $s_p = s_p(f)$  is the slowness, and  $F\{\cdot\}$  and  $F^{-1}\{\cdot\}$  denote the Fourier Transform operator  
 258 and its inverse, respectively. The obtained waveforms at the 20 stations corresponding to the  
 259 slowness of Figure 1b are illustrated Figure 2, where the duration of the wave train with distance  
 260 can be appreciated.

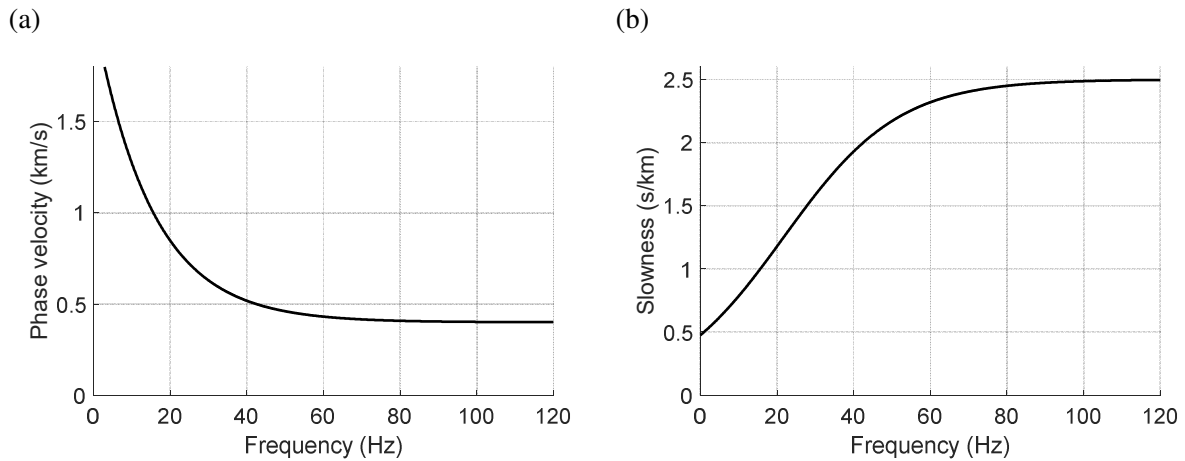


Figure 1. Artificial (a) phase velocity and (b) slowness for single dispersive Ricker wave train.

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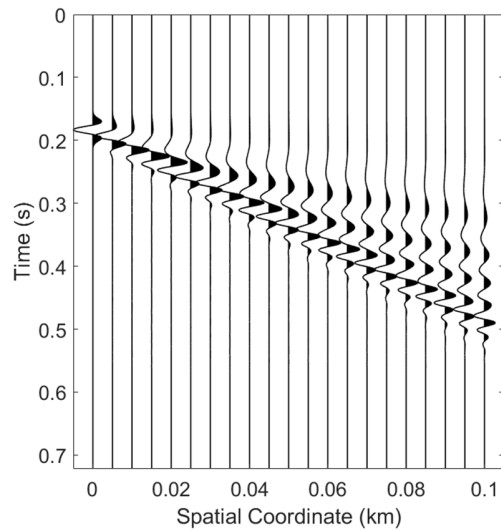


Figure 2. Waveforms at 20 stations for a single Ricker wavelet with artificial dispersion.

262 In order to further illustrate the dispersion effect, in Figure 3a and 3b I show the S-Transform of  
 263 the signals at the first and last stations. The separation of the arrival times of the peaks of the  
 264 amplitude of the S-Transform at the final station is evident, even though the components of the  
 265 waveform are not separated, and at the final station it can be still considered that a single wave  
 266 train is present in the signal. Although not a necessary step in the procedure, in Figures 3c and 3d  
 267 I show the reassigned S-Transforms at the first and last stations using the time delay  $t_d$ , to  
 268 illustrate the sharpening effect of reassignment. After reassignment the peaks of the S-transforms  
 269 can be more precisely identified. With this artificial example I will recover the dispersion curve  
 270 using the seismograms at the first and the final stations of the arrangement, because there is no  
 271 wave interference or loss of energy in the propagated wave. In more realistic settings, interaction  
 272 with other waves and geometrical and material attenuation would require the use pairs of stations  
 273 that are as close as possible.

274

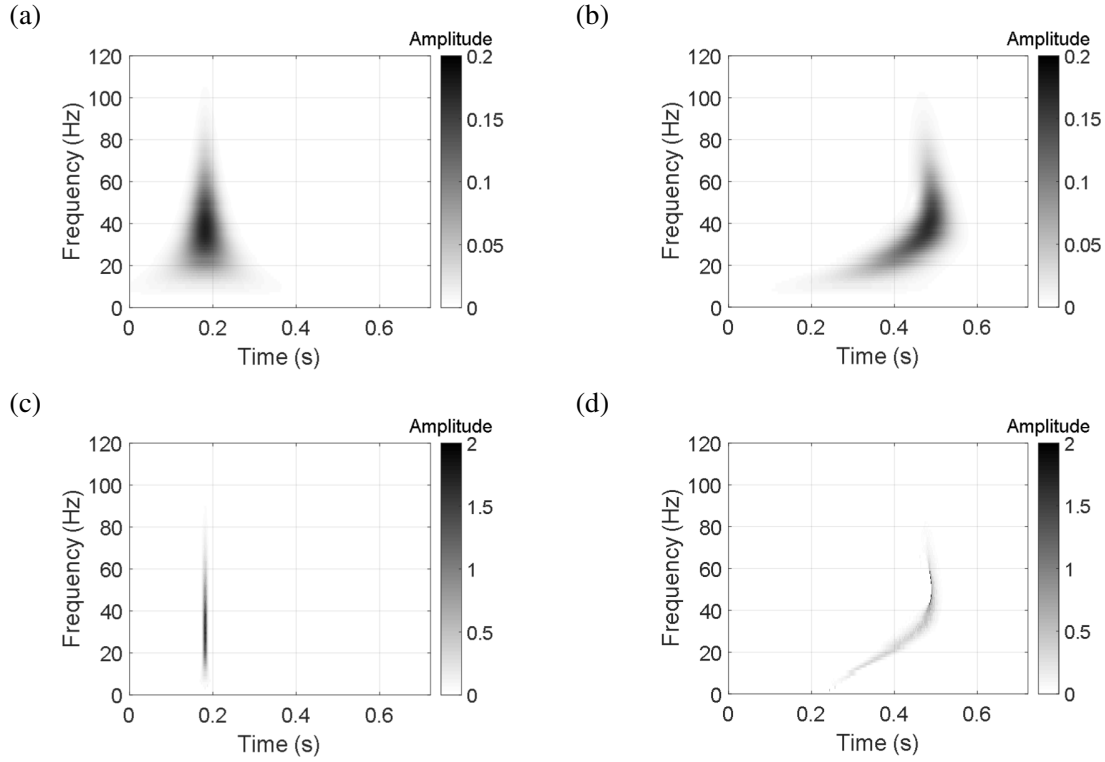


Figure 3. Amplitude of S-transforms of dispersed Ricker wavelet. (a) S-transforms at first station, (b) S-transforms at final station, (c) Reassigned S-transforms at first station (d) Reassigned S-transforms at final station.

275 In order to illustrate the procedure presented in the previous section, I now proceed to compute  
 276 the shifted S-Transform  $S_{2s}$  corresponding to the final station. I apply the same shifts to the  
 277 voices of the transform  $S_{2t}$  because it is also needed to compute the Generalized Time Shift. In  
 278 Figure 4 I plot the results of shifting the transforms  $S_2$  and  $S_{2t}$  to show that indeed their peaks  
 279 have the same location as the peaks of the S-transform of the first station  $S_1$ . No changes in the  
 280 amplitudes are introduced, as the amplitude scales in Figure 4 show. Next, in Figure 5a, I plot the  
 281 amplitude of the Cross-S-Transform  $S_c(t, f_n)$  between the first and last stations. Even though all  
 282 the information to obtain the dispersion curve is given by  $S_c(t, f_n)$ , it is not visually identifiable  
 283 in Figure 5a, and a reassignment  $S_c(t, f_n)$  might be convenient. In Fig. 5b I plot the normalized  
 284 amplitude  $\tilde{A}_{RC}(T_p, f)$  of the Reassigned Cross-S-Transform, which I compute in the following  
 285 manner:

$$\tilde{A}_{RC}(T_p, f) = \frac{A_{RC}(T_p, f)}{\max_{T_p, f} A_{RC}(T_p, f)} \quad (45)$$

286 where  $A_{RC}(T_p, f)$  is the amplitude of  $S_{RC}(T_p, f_n)$ . The vertical axis of the RCST in Fig. 5b  
 287 corresponds then to the time-shift. It can be observed that the concentration of energy of the  
 288 RCST around the time shift is very high, to the point that the non-zero points of the RCST are  
 289 not very visible in Fig. 5b. That is why in Figure 5c, where I compare the estimated slowness  
 290 with the original slowness of Figure 1b (shown with the continuous line), I added black dots to  
 291 make the non-zero points of the RCST more discernable. To obtain the dispersion curve of Fig.  
 292 5c, I simply divided the time shift axis of the RCST by the interstation distance, which in this  
 293 case is 0.1 km.

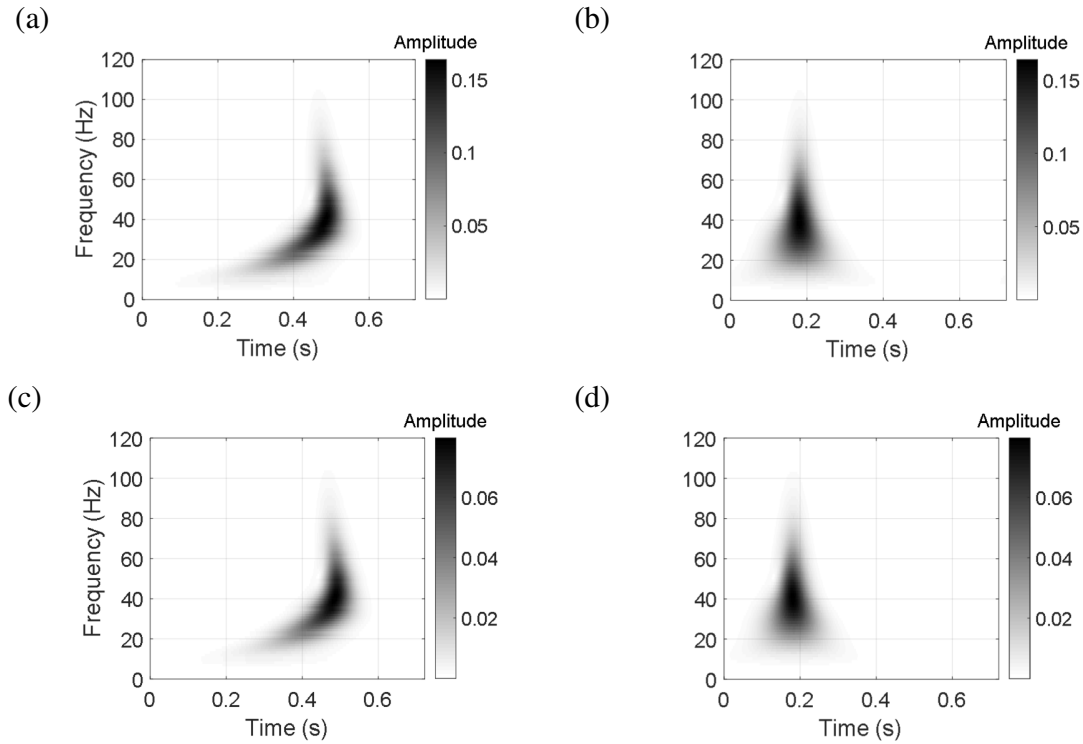


Figure 4. Amplitude of S-transforms of dispersed Ricker wavelet at last station. (a) Regular S-Transform,  $S_2$ , (b) Shifted S-Transform,  $S_{2s}$ , (c) S-Transform  $S_{2t}$ , (d) Shifted S-Transform  $S_{2st}$ .

294



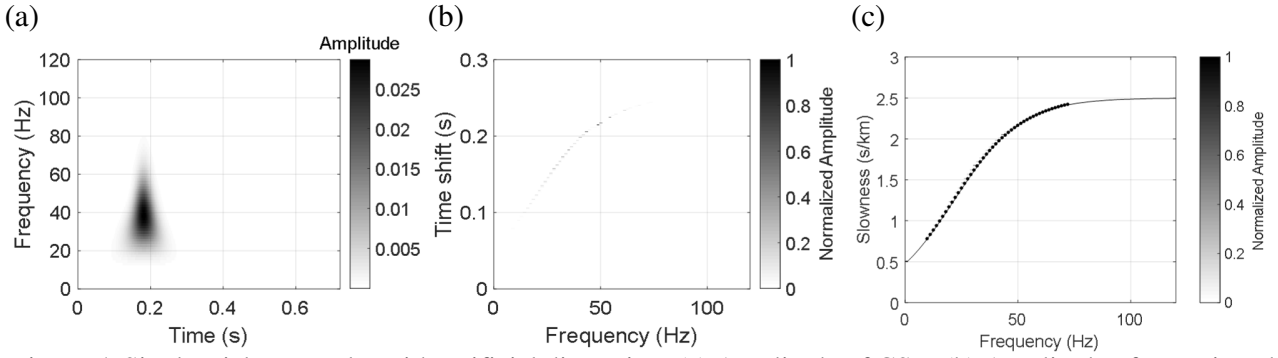


Figure 5. Single Ricker wavelet with artificial dispersion. (a) Amplitude of CST, (b) Amplitude of Reassigned CST (c) Dispersion curve compared with the original slowness curve (the continuous line).

296 Figure 5c illustrates that with the RCST very precise estimations of the slowness curve can be  
 297 obtained. However, for signals composed of a non-contaminated single wave train, the  
 298 computational cost of estimating the RCST is not justified, as simple Fourier analysis as in [20]  
 299 provides very accurate results for the time shift. One advantage that the RCST over other  
 300 methods of analysis is that its non-zero values are associated to the energy of the wave train  
 301 present in the signal, and therefore, discerning the physical wave trains from noise or interfering  
 302 waves becomes easier. As it can be observed in Figure 5c, the RCST gives values for slowness  
 303 only within the frequency band of the wave train.

### 304 6. Example with two artificial dispersive wave trains propagating in the same direction

305 In this example I add another wave train to the signals, with a different artificial dispersion  
 306 curve. In this way I intend to simulate the propagation of bi-modal dispersive waves. I add  
 307 another Ricker to the first mode I already analysed, with a lower amplitude  $A_{R2} = 0.5$ , central  
 308 frequency  $f_{c2} = 60 \text{ Hz}$ , and  $t_{c2} = 1/f_{c2}$ . For this new “higher mode” the slowness law is shown  
 309 in Figure 6a with the thicker line, and the mathematical expression for the phase velocity is

$$V_{p2} = 4 \exp\left(-\frac{f}{17}\right) + 0.55 \left(\frac{km}{s}\right) \quad (46)$$

310 I adopt again 20 stations spaced at 5 m to propagate the wavelets, the resulting waveforms are  
311 shown in Figure 6b. From the signals of Figure 6b it is not evident if the two propagated modes  
312 are separated in time. However, from inspection of the S-transforms of the first and last stations,  
313 shown in Figures 7a and 7b, it is clear that the two modes are mixed at the first station, but  
314 separated at the final station. Thus, the strategy I follow here to compute the dispersion curve is  
315 different from the previous example. Instead of only using the first and last stations to compute  
316 the dispersion curve, I compute the dispersion curve between each pair of adjacent stations, and  
317 average the result after normalizing each dispersion curve according to Eq. (45). The final  
318 average dispersion curve is shown in Figure 7c, and compared with the original slowness curves.  
319 Even though the dispersion curve of second mode is less energetic, is correctly identified in the  
320 RCST. At about 50 Hz, the dispersion curve for the first mode moves away from the original  
321 slowness curve, and some spread of energy can be observed. This is indeed related to the mixture  
322 of the two modes, and one approach to improve the results would then be the analysis at stations  
323 farther from the source, where the two modes might be better separated. An alternative way to  
324 reduce this effect is to separate the two modes in the seismogram, before the dispersion analysis.  
325 However this solution requires measurements of at least two components (horizontal and  
326 vertical) of the waveforms at each station, usually unavailable in geophysical measurements,  
327 which retain only the vertical component. I will explore further this approach in the example of  
328 section 8.

329



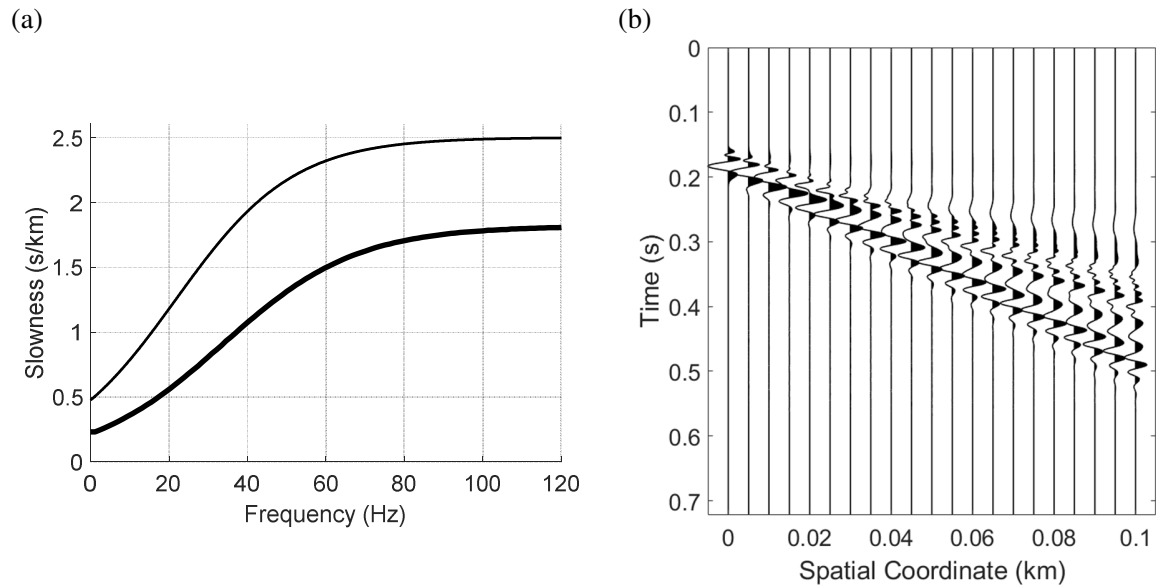


Figure 6. Artificial bi-modal dispersive Ricker wave train. (a) Slowness function, (b) Waveforms of propagated waves at 20 stations.

330

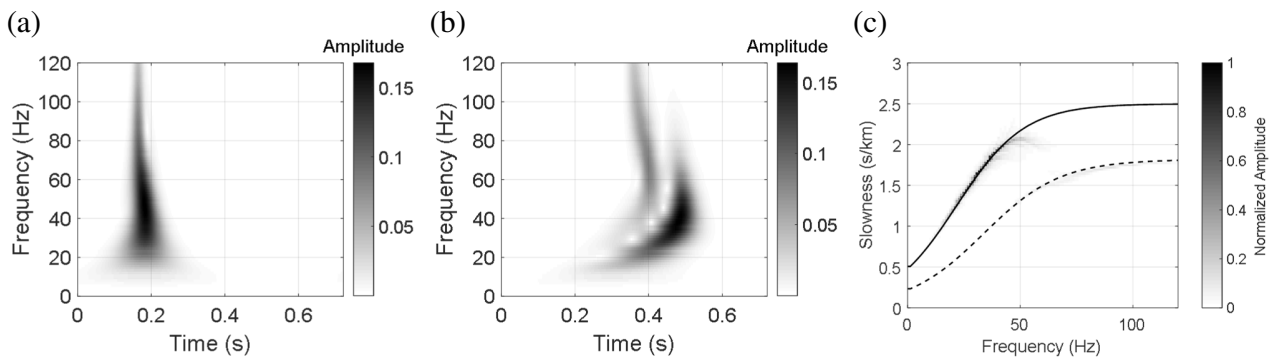


Figure 7. Bi-modal Ricker wavelet with artificial dispersion. (a) S-Transform at first station, (b) S-Transform at last station, (c) Average dispersion curve compared with the original slowness curve (the continuous line).

331 **7. Example with two artificial dispersive wave trains propagating in opposite**  
 332 **directions**

333 In this experiment I consider the same two Ricker wave trains (modes) used in the previous  
 334 example, but now the weaker wave train propagates in the opposite direction. With this  
 335 experiment I try to simulate the presence of reflected/refracted waves in the signals, and  
 336 investigate the effect on the estimated dispersion curve. The time histories of the waveforms are  
 337 shown in Figure 8a, where it can be observed that the interference between the wave trains starts

338 at 0.06 km. In Figure 8b I present the average dispersion curve, which I obtained in the manner  
 339 explained in the previous example. One effect that can be observed in Figure 8b is the  
 340 “spreading” of the energy due to the interference between the wave trains. A weak presence with  
 341 negative slope is also noticeable in Figure 8b, and how it diverts the dispersion curve after 50  
 342 Hz. In situations like this example, separation of the dispersive waves from the  
 343 reflections/refractions can improve the accuracy of the slowness curve. However, the analyst  
 344 should also keep in mind that the presence of those reflections/refractions might indicate  
 345 lateral/horizontal heterogeneities in the velocity structure.

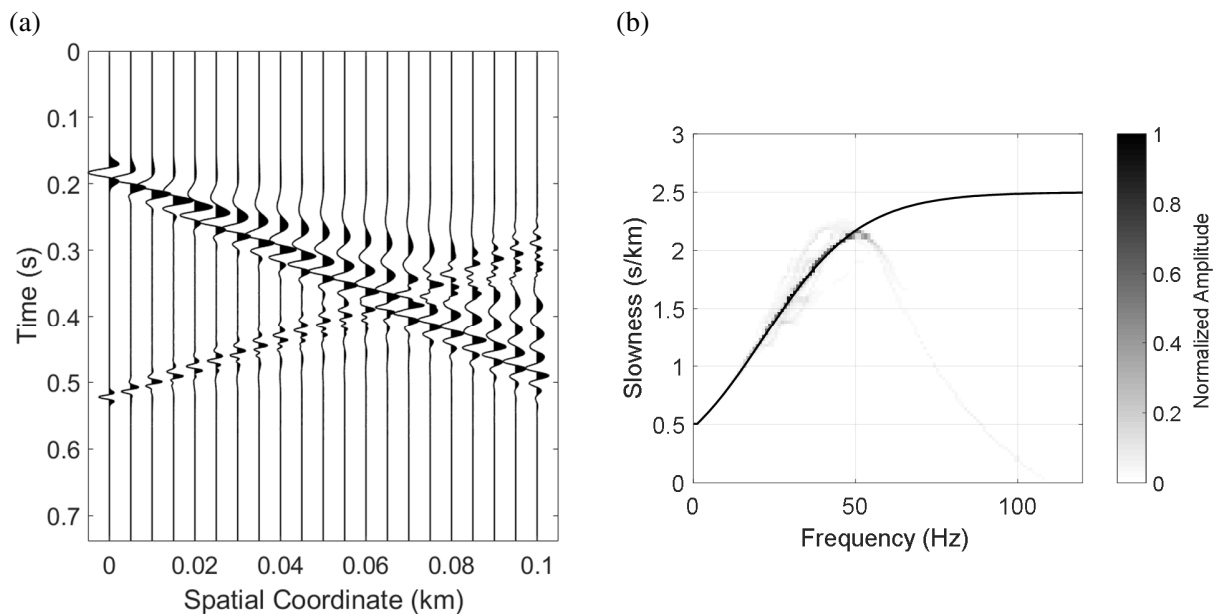


Figure 8. Artificial dispersive waves propagating in opposite directions. (a) Waveforms at 20 stations, (b) Average dispersion curve compared with the original slowness curve (the continuous line).

### 346 **8. Example of mode separation from numerical simulations**

347 In this last example I investigate the benefits of mode separation in the estimation of the  
 348 dispersion curves. An efficient technique based on the S-Transform to identify and separate wave  
 349 trains in seismograms has been proposed in the recent years [8], which provides the time  
 350 histories of the extracted waves. However, the extraction technique is based on the polarization

351 characteristics of the waves and therefore it requires the vertical component and at least one  
 352 horizontal component of the recorded/generated time histories, while most near-surface surveys  
 353 retain only the vertical component. As it is the case in MASW surveys, the seismic source is a  
 354 vertical force at the free surface of an earth model. In order to simulate a wave field with  
 355 geometrical dispersion, the selected earth model is composed of several layers with different  
 356 density and elastic constants. I adopt the earth model described in Table 1, which was already  
 357 used by Xia *et al.* [16] in numerical simulations for parametric studies on inversion of velocity  
 358 profiles from the fundamental mode of Rayleigh waves. With the mechanical parameters listed in  
 359 Table 1 I then compute the Rayleigh wave theoretical dispersion curve with the software  
 360 Gplivemodel [21], a Geopsy tool to compute dispersion curves for layered 1D earth models  
 361 using the propagator-matrix method [23]. Figure 9 shows the resulting theoretical slowness  
 362 curves for Rayleigh waves, and that around 15 Hz the curves of the fundamental and first higher  
 363 mode are in close proximity.

Table 1. Mechanical parameters of layered earth model.

Layer	$v_s$ (m/s)	$v_p$ (m/s)	$\rho$ ( $g/cm^3$ )	Thickness (m)
1	194	650	1.82	2.0
2	270	750	1.86	2.3
3	367	1400	1.91	2.5
4	485	1800	1.96	2.8
5	603	2150	2.02	3.2
Half-space	740	2800	2.09	

364

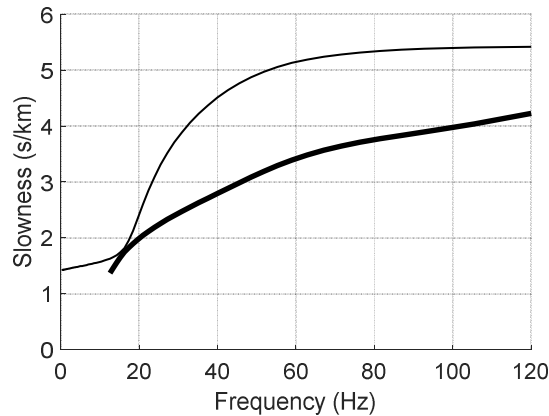


Figure 9. Theoretical slowness curves (fundamental and first higher mode) for Rayleigh waves in the earth model of Table 1. The thicker line corresponds to the higher mode.

365

366 To propagate Rayleigh waves I limit a rectangular region of the earth model to have 160 m of  
 367 length and 36 m of total depth, so that all layers described in Table 1 are considered. I then  
 368 discretize the 2D P-SV case of the wave equation for this region, using the Spectral Element  
 369 Method (SEM) [24]. The square elements have a size of 2x2 m, and the degree of the interpolant  
 370 of the field variables is 4. The top boundary of the 2D spatial region corresponds to the free  
 371 surface, whereas the other three boundaries have to comply with the condition of energy radiated  
 372 to the infinity, as illustrated in Figure 10a. To enforce such radiation condition I implement the  
 373 Multi-axial Perfectly Match Layer (M-PML) [25], an accurate and stable absorbing boundary for  
 374 isotropic elastic media. Each M-PML termination has 10 elements parallel to the truncated  
 375 boundary. The discretization in time of the equation of motion is made with the midpoint rule  
 376 with a time step  $\Delta t=4.93E-05$  s. I use again a Ricker wavelet with a central frequency of 20 Hz  
 377 to describe the time-variation of the source. The source, imposing a vertical force at the surface,  
 378 is located at 10 m from the first receiver. The interstation distance for the receivers is 2 m. The  
 379 resulting time histories of the vertical and horizontal displacement at 41 receivers on the free

380 surface are shown in Figure 10b and 10c, respectively. The propagation of the two first modes of  
381 Rayleigh waves is clearly distinguished along the stations.

382 I select the vertical component of the waveforms to estimate the dispersion curve, usually  
383 preferred to avoid the presence of other dispersive waves (such as Love waves) in the signal. In  
384 this 2D P-SV simulation Love waves cannot be generated, nevertheless I make the initial  
385 analysis with the vertical component. The resulting slowness curve is shown in Figure 11a,  
386 where I again added some black dots on the non-zero values of the RCST to improve the clarity  
387 of the plot. Even though the simulations seem to be of high accuracy as there is no contamination  
388 due to reflections from the absorbing boundaries, several problems are observed with dispersion  
389 curve of Figure 11a. First, the higher mode is barely detected, because of the small energy  
390 compared to the fundamental mode. And second, at around 15 Hz the higher mode is  
391 misidentified as the continuation of the fundamental mode. In Figure 11b I add the theoretical  
392 curves to compare with the obtained results, where the misidentification problem is more  
393 evident. In real cases, where there are no “theoretical curves” available to the analyst, the mode  
394 misidentification problem can be overlooked, leading to incorrect inverted velocity profiles.

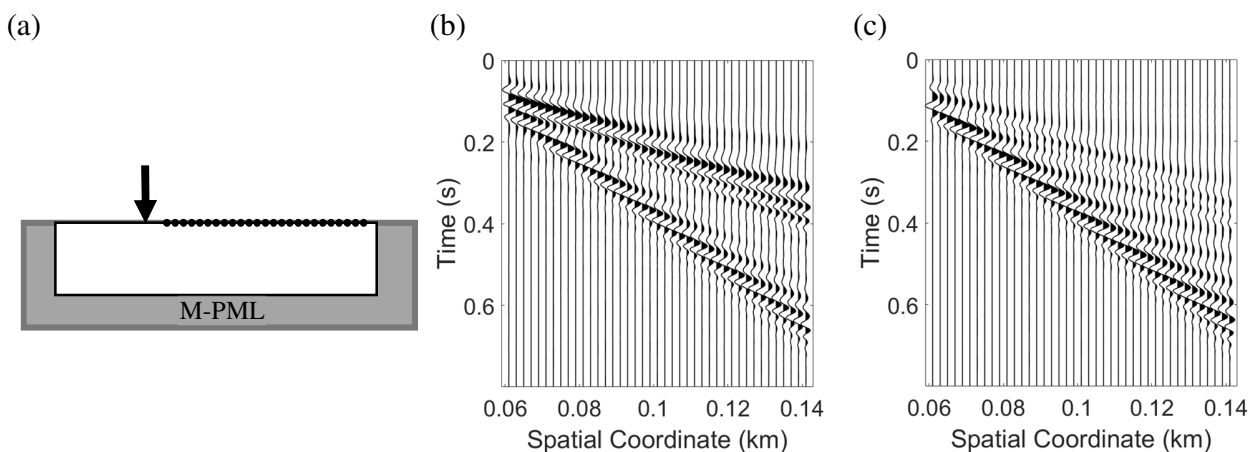


Figure 10. 2D Numerical simulation of seismic wave propagation in a layered medium. (a) Sketch of the propagating medium, with location of the source, receivers and absorbing boundaries. (b) Time histories of horizontal displacements, (c) Time histories of vertical displacements.

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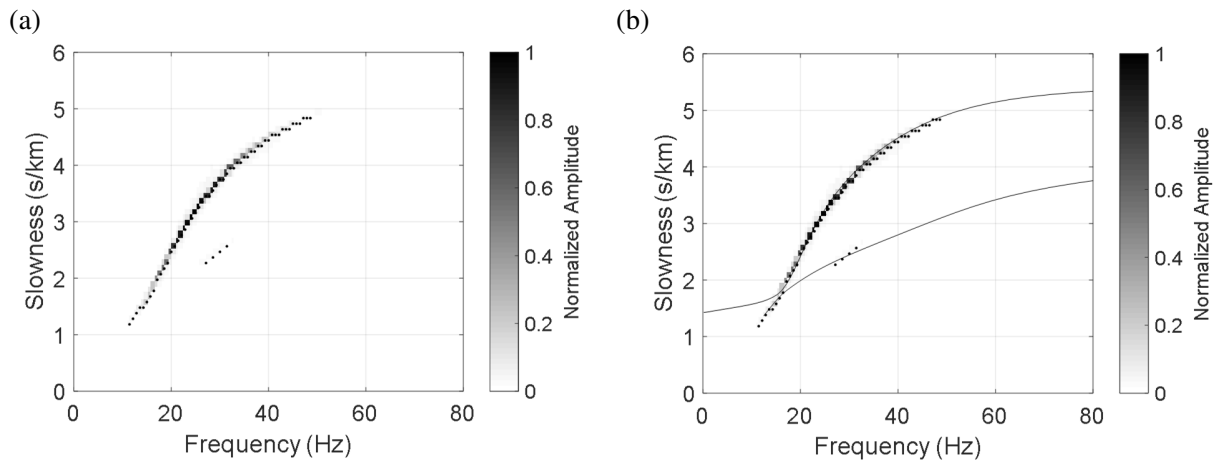


Figure 11. Average slowness curves from vertical component of simulated Rayleigh waves. (a) Dispersion curve from RCST, (b) Dispersion curve from RCST compared with theoretical curves.

396

397 The strategy I propose here to avoid the misidentification problem is to separate the two  
398 Rayleigh modes present in the signals with the *Normalized Inner Product* (NIP). For details on  
399 the application of the NIP technique to multi-component recordings, the reader is referred to the  
400 references [8], [9]. Because the NIP is a time-frequency technique that identifies and extract the  
401 waves according to their polarization characteristics, it efficiently identifies Rayleigh waves,  
402 which are elliptically polarized waves. In most cases the fundamental mode of Rayleigh waves  
403 has a *retrograde motion* at shallow depths, meaning that the particle motion is an elliptical  
404 rotation against the propagation direction. In other cases the fundamental mode and/or the higher  
405 modes can present *prograde motion* at the surface, an elliptical rotation in the same direction of  
406 wave propagation.

407 In Figure 12a I plot the vertical component of the time histories of the retrograde waves extracted  
408 with the NIP, and it is clear that they correspond to the fundamental mode. Fortunately, when  
409 extracting the prograde waves, also with the NIP, I found that they correspond to the less  
410 energetic higher mode of the Rayleigh waves, as shown in Figure 12b; and then the two modes

411 have been successfully separated. The time histories of the separated waves are then used to  
412 compute the slowness curves shown in Figure 13, where the comparison with the theoretical  
413 curves is included. It is evident that after mode separation, the estimated dispersion curves are  
414 very precise, and the mode misidentification problem is resolved.

415

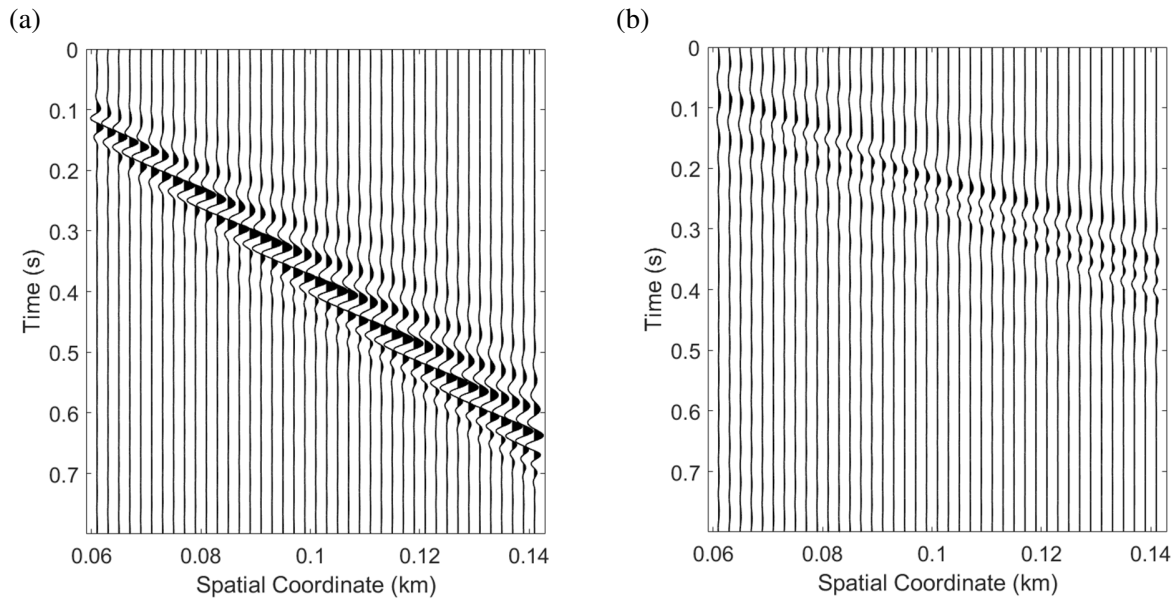


Figure 12. Vertical component of time histories of simulated Rayleigh waves. (a) Fundamental mode (retrograde motion), (b) First higher mode (prograde motion).

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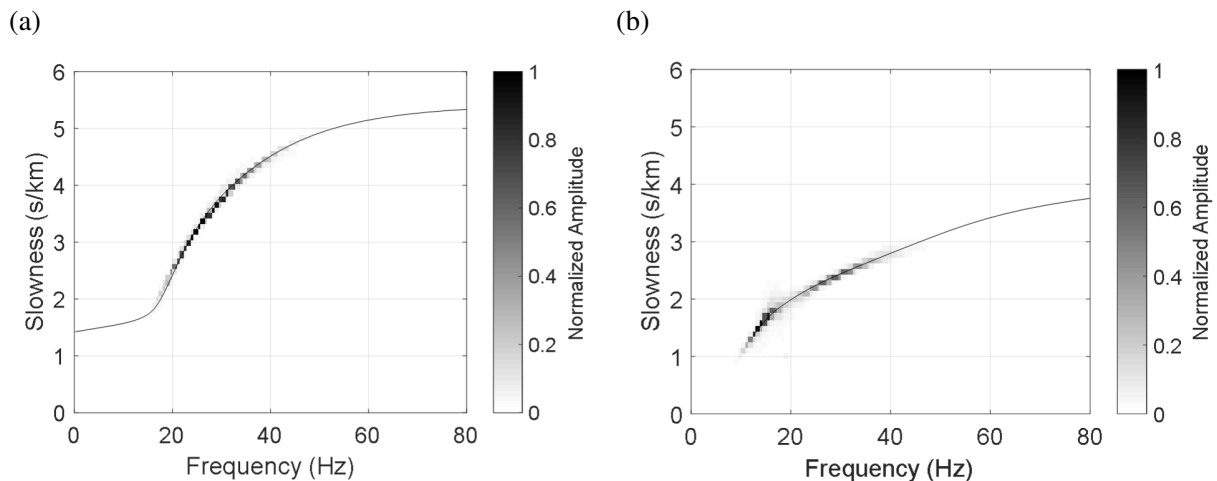


Figure 13. Average slowness curves from vertical component of simulated Rayleigh waves. (a) Fundamental mode, (b) First higher mode.

417 **9. Example with real recordings**

418 In this example, I estimate the dispersion curve via the RCST from actual recorded data which is  
 419 provided with the software Geopsy [21], an open source tool for geophysical research and  
 420 applications (*e.g.*, [22]). The 1D linear arrangement of seismic surface wave recordings is to be  
 421 analyzed with the active seismic experiment (MASW) tool implemented in Geopsy, based on the



422 F-K transform. The profile of recorded waveforms is presented in Figure 14, where for clarity I  
423 normalized each time history by its maximum amplitude. The active source is located at 4 m  
424 from the first sensor, and the interstation distance of the 24 sensors is also 4 m. The time step of  
425 the time histories  $\Delta t = 2.5E-4$ . Figure 14 clearly shows that there are at least two modes  
426 propagating through the different stations, as well as other wave types interfering with the  
427 modes.

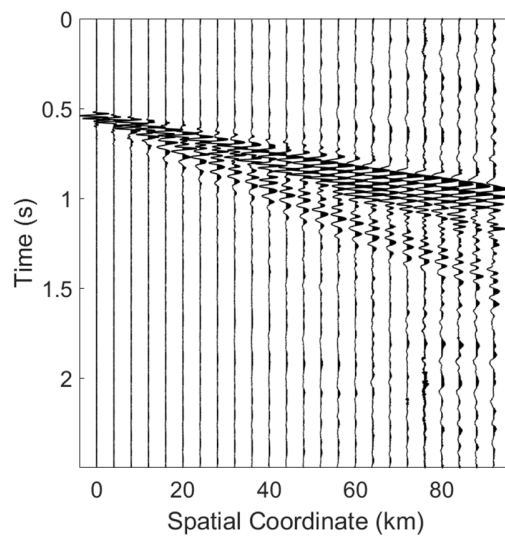


Figure 14. Recorded waveforms at 24 stations in a MASW survey.

428 In Figure 15 I compare the slowness curves obtained with the RCST with the results given by the  
429 active MASW tool of Geopsy. For clarity, I added the black dots to the peaks of the RCST. The  
430 figures illustrate that the two methods give similar results, and they both identify well the higher  
431 mode, whereas the fundamental mode is well defined up to 20 Hz. The RCST gives shaper  
432 results in more limited frequency bands, because the RCST gives non-zero values only at  
433 frequencies with relatively high concentration of energy. Furthermore, considering the analysis  
434 of the previous artificial examples, the RCST of Figure 15a indicates that the fundamental mode  
435 is interfered not only by the higher mode and noise, but also by some reflected/refracted waves.  
436 This observation needs to be taken into consideration if the dispersion curve will be used to

437 invert a velocity profile, whether those interfering waves are related to the physical process or  
438 not.

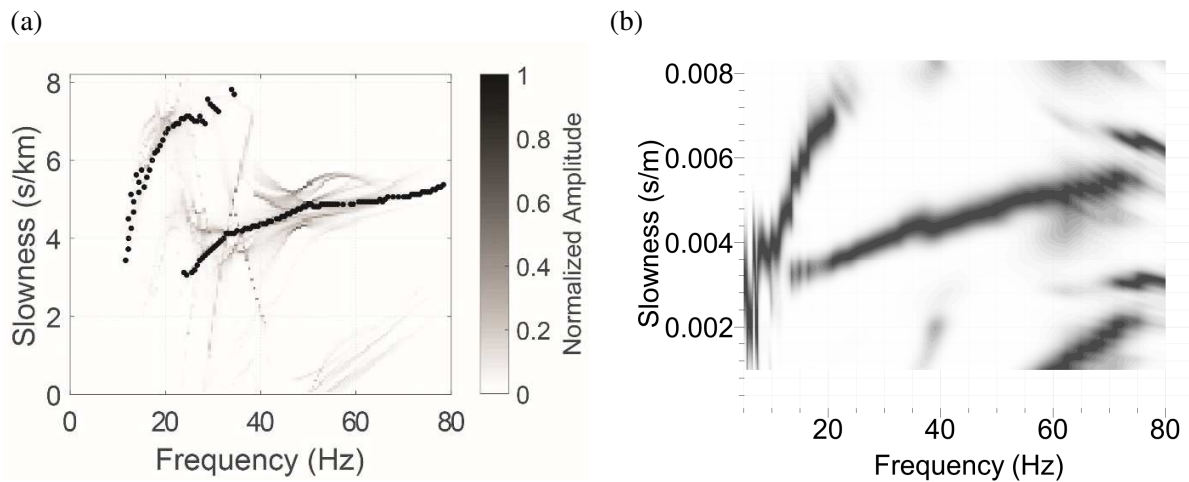


Figure 15. Slowness curves of recorded waveforms. (a) Average dispersion curve from RCST, (b) Dispersion curve from Geopsy software.

## 439 10. Conclusions

440 In this work, I proposed a technique to estimate dispersion, slowness curves for multimodal, non-  
441 stationary signals, based on the Reassigned Cross-S-Transform. I presented in detail the  
442 mathematical derivations and the computational steps for the construction of the RCST. The  
443 RCST I proposed is derived from simple maximization principles so that the energy is  
444 concentrated at the ridges of the time-frequency representations of the signals. Furthermore, in  
445 deriving the RCST I made no assumptions regarding the types of waves or the cinematics of the  
446 physical process. The objective of the proposed reassignment is twofold (i) improve the  
447 resolution of the time-frequency representation, and (ii) concentrate the energy of two signals in  
448 the neighborhood of the time-shift related to directly provide the phase velocity. **An additional**  
449 **advantage of the procedure I propose is that because the time shift between signals is obtained**  
450 **from cross-correlation, no time readings are needed, and dispersion curves can be computed for**  
451 **data that is not synchronized with the same initial time.** I illustrated the application of the RCST

452 with seismic signals containing dispersive waves, constructed from artificial dispersion curves,  
453 generated via 2D numerical simulation, and recorded in a real survey. The examples with multi-  
454 modal artificial waves allowed to recognize the effects on the dispersion curve of modes  
455 overlapping in time and frequency, and the effects of possible reflections/refractions present in  
456 the signals. Regardless of the identified detrimental effects, the dispersion curves of the different  
457 artificial modes were well identified with the RCST. The analysis is then helpful to interpret the  
458 dispersion curves from real seismic recordings containing multi-modal Rayleigh waves, for  
459 which the RCST provides very sharp dispersion curves. Furthermore, I showed that combining  
460 the RCST with the NIP technique of extraction of dispersive waves is an effective strategy to the  
461 correct identification of modes, in particular when the dispersion curves are in close proximity.

## 462 **11. Acknowledgements**

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